OBTAINING DEFICIENT INFORMATION BY SOLVING INVERSE PROBLEMS FOR MATHEMATICAL RUNOFF MODELS

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ABSTRACT

Possibilities are considered for increase of deficient information for extending observation series by solving the "inverse problem" for mathematical runoff models. The results of applying the theory of "improperly posed problems" are presented. Examples are given for representing hydrological, geometrical and hydraulic characteristics of the basin by lumped and distributed parameter runoff models.

RESUME

Les auteurs examinent les possibilités de la resolution du problème inverse appliquée aux modèles mathématiques d'ecoulement, en vue de compléter les lacunes des séries d'observations et d'étendre la période couverte par ces séries. Ils exposent les résultats qui ont été obtenus par l'application de la théorie des problèmes posés incorrectement. Ils citent des exemples de détermination des caractéristiques hydrologiques, topographiques et hydrauliques a l'aide de modèles d'ecoulement globaux ou matriciels.

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Mathematical modelling of hydrological processes is increasingly used to provide for missing information and to extend hydrological time series. Mathematical models are predominantly used for the solution of the so-called 'direct problem', consisting of derivation of unknown hydrological variables by solving respective differential equations with known coefficients and known initial and boundary conditions. In a large number of cases it is necessary to solve the 'inverse problem' namely to find the coefficients and establish the initial and boundary conditions using observed values of the hydrological variables included in the equations. This approach has as yet gained relatively rare use due to the fact that the solution of the 'inverse problem' is more difficult than that of the 'direct problem'. The solution of the 'inverse problem' may be circumvented by multiple solutions of the 'direct problem' for example by the methods of trial and error and subsequent optimization. This may lead however to a non-unique or inferior solution. The principal difficulty in the solution of the inverse problem consists in the fact that it may be incorrectly posed and thus leads to the non-existence of some or any initial conditions or leads to a solution in which a small change of initial conditions (data) due for example to observational errors, results in major changes in the results. This has caused in the past a reluctance toward the use of this method, since the solution being of very low accuracy and high uncertainty casts doubt on its physical significance.

A number of studies were made in recent years (particularly by A.N. Tikhonov and his school) aiming at the correct posing of the problem by establishing the necessary conditions for it. A.N. Tikhonov has shown that it is possible to use a priori information on the solution to ensure a continuous dependance of the solution of an incorrectly posed problem on its initial conditions and to derive special algorithms which prevent bringing out the solution outside the limits of its uniqueness and of the existence of its initial conditions. In particular it made possible to solve with sufficient stability such classical incorrectly-posed problems as the integral equation of the first type, algebraic systems with improper initial conditions, the Cauchy problem of the Laplace equation and others. The theory of the 'inverse problem' has thus stimulated the formulation of algorithms used in many scientific and technical fields. The method was particularly useful in geophysics, where it permitted the solving, for example, of problems of determination of rock characteristics not accessible for direct measurement as well as restoration of missing information, to cite only the most important points. The use of this method in hydrology appears also as most promising. Examples of such studies, used in hydrological practice, are given below. They illustrate also the principles and possibilities of the theory of incorrectly posed problems.

1. Determination of the input functions of the models with lump parameters

Let us suppose that the process of transforming an input h(t) in the catchment (effective rainfall or an inflow) into an output Q(t) can be described by the Duhamel integral:

$$Q(t) = \int_0^t P(t-\tau)h(\tau)d\tau \qquad (1)$$

where $P(t)$ is some known function of influence. Then having the observations on $Q(t)$ and knowing the function $P(t)$ (by historic observations or from physiographic and hydraulic data) it is possible using (1) to derive $h(t)$. Thus an improperly posed problem is solved - consisting of an integral equation of the first type. It is possible to solve this problem on the basis of A.N. Tikhonov's algorithm. Integral (1) is replaced by a summation according to the method of rectangles and a smoothed functional curve is constructed:

$$\Phi[A, \vec{P}, \vec{h}] = \|A\vec{h} - \vec{Q}\|^2 + \alpha\|\vec{h}\|^2 \qquad (2)$$

where $\vec{Q}$ = a vector, designating the ordinates of the given hydrograph $Q(t)$; $\vec{h}$ = a vector of the unknown ordinates h_j; A = a matrix with elements P_{i-j+1}; α = a positive constant. Finding the minimum of this functional makes it possible to receive a sequence of stable solutions $\vec{h}$, which converge to the accurate solution providing there are no errors in the given data. However since there are always errors in these, changing the parameter α (called parameter of regularization) we select such solution which corresponds best to the <u>a priori</u> information about the function $h(t)$. For example good results are obtained with the aid of the condition $\int_0^{\infty} h(t)dt = \int_0^{\infty} Q(t)dt$.

Other kinds of <u>a priori</u> information, allowing the narrowing of the interval of unknown solutions, may be a suggestion on the smoothness of the solution, the non-negativeness of the ordinates, the closeness to some known function and so on. Naturally, the narrower the interval of the solution, the higher accuracy will be obtained. Results in using functional (2) to determine the input functions of the runoff models, described by the Duhamel integral, are presented in greater detail in (3), where examples of constructing effective rainfall, hydropower station releases and snowmelt intensity are treated. Another approach to the solution of the inverse problems for models described by the Duhamel integral (linear models with lump parameters) are indicated in (6).

2. <u>Determination of geometric and hydraulic characteristics of river channels using observations of flow</u>

To describe unsteady flow in a river channel Saint Venant equations may be used:

$$
\begin{aligned}
-\frac{\partial z}{\partial x} &= \frac{Q^2}{K^2} + \frac{1}{g}\frac{\partial}{\partial t}\left(\frac{Q}{F}\right) + \frac{1}{2g}\frac{\partial}{\partial x}\left(\frac{Q^2}{F^2}\right) \\
\frac{\partial Q}{\partial x} &+ \frac{\partial F}{\partial t} = 0
\end{aligned}
\qquad (3)
$$

where Z (x,t) = stage at point x at time t, Q(x,t) = discharge, K(x,z) forces of resistance; g= acceleration of gravity. Because of great variability of geometry and roughness of the river channels the functions F(x,z) and K(x,z) determined by the observations in separate points are not quite representative for the whole river reach, even with large frequency of observations. Thus a problem of determining the averaged relations P(x,z) or B(x,z) = ∂F/∂Z and K(x,z) by observations of flow (the determination of coefficients of the system (3)) is of great significance for the establishment of the most characteristic geometry and hydraulic properties of the river channel as well as for ensuring sufficient accuracy of the calculations. It can be shown that this problem is improperly posed and for its solution it is necessary to derive special calculating algorithms. We shall discuss below two of the approaches tried by us in solving this problem.

(A) The discharges and water levels are known in a rather large number of sites.

Integration of the continuity equation (3) with respect to x, leads to:

$$Q(x,t) - Q(0,t) = -\frac{\partial}{\partial t}\int_0^x F(\eta,t)\,d\eta \qquad (4)$$

Finite differences are substituted for the derivatives and instead of an integral it is possible to construct for every time moment j the following system of equations:

$$\sum_{k=0}^{i}\{F(j+1,k)+F(j+1,k+1)-F(j,k)-F(j,k+1)\} = \frac{\Delta t}{\Delta x}[Q(j+1,0)+Q(j+1,i)-Q(j,i)-Q(j,0)]_j \qquad (5)$$

$$(i=1,2,3,\dots,N)$$

or in matrix form: $A\vec{F}=\vec{Q}$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 2 & 1 & 0 & \cdot & & \cdot & 0 \\ 2 & 2 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & & 2 & \cdots & & \cdot & 1 \end{bmatrix}$$

In order to solve this system it is necessary to have Q(x,t) F(x,o) and F(o,t). As the problem is improperly posed the solution of the system (5) is unstable. For its regularization the solution of A.N. Tikhonov's functional is with introducing initial approximation. As a result for every time such F are found which correspond to the minimum of the functional.

$$\Phi^{\alpha}[\vec{F}] = \|A\vec{F}-\vec{Q}\|^2 + \alpha\|\vec{F}-\vec{F}_0\|^2 \qquad (6)$$

where $\vec{F}_o$ is the given initial approximation, α = positive parameters, thus a solution is found which not only secures the minimum of square deviation of the right part of the system (5) from the left part, but at the same time it is least deviated from the initial approximation. The condition of functional extreme gives:

$$(A^*A+\alpha E)\vec{F} = A^*\vec{Q}+\alpha E\vec{F}_o \qquad (7)$$

To select the quantity α method of discrepancy has been used. The idea of this method consists in conforming the accuracy of the problem's solution to the accuracy of observed data.

It is supposed that the errors of the given information forming discrepancy of the system (5) are known and an α is found which secures this discrepancy δ^2. It is possible to prove that if the functional (6) is used the parameter α securing the given discrepancy is unique. The initial approximation can be made in a rather crude manner (particularly for $\vec{F} = 0$), however giving a good initial approximation contributes to a more accurate optimum α. Use of the initial approximations provides great possibilities for improvement of the solution by introduction of <u>a priori</u> information. Such <u>a priori</u> information can be an empirical relationship between geometrical and hydraulic characteristics, observed in separate sites, and different theoretical formulas (for example, we have used the equation of the typical form of river channel derived from the principle of minimum dissipation of energy).

The values of F (x,t) found according to equation (4) have been used for determining the characteristics of the resistant forces. For this purpose the momentum equation has transcribed:

$$Z(0,t)-Z(x,t)=\int_0^x\frac{Q^2}{K^2}d\eta+\frac{1}{2g}\left[\frac{Q^2(x,t)}{F^2(x,t)}-\frac{Q^2(0,t)}{F^2(0,t)}\right]+\frac{1}{g}\frac{\partial}{\partial t}\int_0^x\frac{Q}{F}d\eta \qquad (8)$$

Derivatives with respect to t have been replaced by forward directed finite differences and the integrals have been replaced by sums derived by the method of rectangles. The resulting algebraical systems have been solved for all time intervals with the help of the same algorithm as the system (5) (without the initial approximation).

As for determining F(x,t) and K(x,t) the discrepancy has been taken equal to 5 per cent of the average module from the left integral equation's part.

This method has been tested on data obtained by special observations of unsteady movement in the Tverca river and it has given satisfactory results (a comparison has been made between the relations F(x,z) and K(x,z) which have been derived by different floods by measurements in separate sites) (see figure 1).

(B) The stages are known in a rather great number of sites and the discharges only in the first and the last site.

Let us integrate the continuity equation with respect to time (in the interval (T_i, T_{i+1})) and to distance (in the interval (0, L)):

$$\int_0^L \{F(x, T_{i+1}) - F(x, T_i)\} dx = -\int_{T_i}^{T_{i+1}} \{Q(L,t) - Q(0,t)\} dt \qquad (9)$$

to solve it in the form:

$$F(x, z) = \sum_{k=0}^{n_1} \sum_{s=0}^{m_1} A_{ks} \Psi_k(x) \Psi_s(z) \qquad (10)$$

where Ψ - the Chebishev polinomials. Let us put (10) in (9):

$$\sum_{k=0}^{n_1} \sum_{s=1}^{m_1} A_{ks} \int_0^L \Psi_k(x) \{\Psi_s[z(x,T_{i+1})] - \Psi_s[z(x,T_i)]\} dx = -\int_{T_i}^{T_{i+1}} \{Q(L,t) - Q(0,t)\} dt. \qquad (11)$$

No terms with zero polinomial are in the left part of the equation (11), because in this case the integral would be equal to zero. The equation (11) is therefore not sufficient for the full determination of the function (10). However it can be used for determining the function B(x,z), which can be presented:

$$\sum_{k=0}^{n_1} \sum_{s=1}^{m_1} A_{ks} \int_0^L \Psi_k(x) \{\Psi_s[z(x,T_{i+1})] - \Psi_s[z(x,T_i)]\} dx = -\int_{T_i}^{T_{i+1}} \{Q(L,t) - Q(0,t)\} dt. \qquad (12)$$

To find the coefficients A_{ks} we shall construct a system of equations (their number must not be less than $M = (n_1+1)m_1$), and change the limits of the integration with respect to time in (11) so as to embrace the whole amplitude of variation of discharges and of stages on the rising as well as on the falling, part of the hydrograph. Let us write this system in the matrix form:

$$\varphi \vec{G} = \vec{x} \qquad (13)$$

Here φ is the matrix of n-th order, its elements are equal

$$\varphi_{ij} = \int_0^L \Psi_k(x) \{\Psi_s[z(x,T_{i+1})] - \Psi_s[z(x,T_i)]\} dx$$

$$k = j - 1 - (n_1+1)\cdot ent\left[\frac{j-1}{n_1+1}\right], \quad s = 1 + ent\left[\frac{j-1}{n_1+1}\right]$$

$$(i = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, (n_1+1)\cdot m_1)$$

$\vec{G}$ - vector of the unknown coefficients A_{ks}, x - right part with elements:

$$x_i = \int_{T_i}^{T_{i+1}} \{Q(0,t) - Q(L,t)\} dt \tag{14}$$

Since the system (13) is unstable, its solution is possible with A.N. Tikhonov's functional. As a result the following system is found:

$$(\Phi^* \Phi + \alpha E)\vec{G} = \Phi^* \vec{X} \tag{15}$$

where Φ^* - matrix transformed with relation to Φ, E - the unit matrix.

The parameter of regularization α has been determined from the conditions of minimum of the function

$$K = \sum_{j=1}^{M} [G_j(\alpha_{p+1}) - G_j(\alpha_p)]^2 \tag{16}$$

where $A_j(\alpha_{p+1})$, $A_j(\alpha_p)$ - j-th elements for the two successive values α. For determining $(n_i + 1)$ coefficients entering in (10) we shall replace in (9) discharge with the product of a cross section area and the velocity of the current U(x,t) and shall make the proper integration with respect to time and to distance. Putting in the resulting equation the relation (10) we shall find:

$$C\vec{G}_1 = (\Phi + \Phi_1)\vec{G} \tag{17}$$

Here C - matrix of (nI + I) x N-th order with elements

$$C_{ij} = \int_{T_i}^{T_{i+1}} \{U(0,t)\psi_{j-1}(0) - U(L,t)\psi_{j-1}(L)\} dt \qquad (i=1,2,\ldots,N;\ j=1,2,\ldots,n+1)$$

The rest of symbols are the same. System (17) is solved by analogy with system (13). Having determined $\vec{G}$ and $\vec{G}_1$ it is possible, using relations (10) and (12) to find function B(x,z). This approach has been tested on the data of special observations in the Svir river. In figure 2 functions B(x,z) for some sites, calculated by relation (10) are shown: furthermore widths were determined according to topographic data. For controlling the results of these calculations the discharges in the intermediate sites have been determined with the help of equation:

$$Q(x,t) = Q(0,t) - \int_0^x B(x',z)\frac{\partial z}{\partial t} dx' \tag{18}$$

These discharges have been found as very close to those observed. The coefficients A_{ks} received from the different floods have turned out to be quite similar and this fact indicates their sufficient stability. Let us see now a scheme for determining the hydraulic characteristics of river channels. We use the dynamic St. Venant equation, assuming that the inertial terms are equal to zero

$$-\frac{\partial Z}{\partial x}=\frac{Q^2}{K^2}. \tag{19}$$

Putting (18) into (19) and integrating with respect to distance in the interval (0,I) we get:

$$\int_0^L \frac{1}{K^2}\Big[Q(0,t)-\int_0^x B(\eta,z)\frac{\partial Z}{\partial t}d\eta\Big]^2 dx = Z(0,t)-Z(L,t). \tag{20}$$

As earlier we shall find the solution in the form:

$$\frac{1}{K^2}=\sum_{k=0}^{n_2}\sum_{s=0}^{m_2} D_{ks}\Psi_k(x)\psi_s(Z). \tag{21}$$

Putting (21) into (20) and using Tikhonov's functional by analogy with the previous one we construct the system of equations for determining the coefficients D_{ks}:

$$\begin{gathered}(\Phi^*\Phi+\alpha E)\vec{D}=\Phi^*\vec{\Delta}\\ \min\sum_{j=1}^{n_2\cdot m_2}[D_j(\alpha_{p+1})-D_j(\alpha_p)]^2\end{gathered} \tag{22}$$

where $\vec{D}$ - vector of unknown coefficients, $\vec{\Delta}$ - vector with elements $\delta_i = Z(t) - Z(t)$, φ - matrix of $(n_{2+1})\cdot(m_{2+1})$ N-th order with elements

$$\varphi_{ij}=\int_0^L\Big[Q(0,T_i)-\int_0^x B(\eta,T_i)\frac{\partial Z(\eta,T_i)}{\partial t}d\eta\Big]^2\Psi_k(x)\psi_s[Z(x,T_i)]dx$$

$$k=j-1-(n_2+1)\,\mathrm{ent}\Big[\frac{j-1}{n_2+1}\Big],\quad s=\mathrm{ent}\Big[\frac{j-1}{n_2+1}\Big]$$

$$(i=1,2,\dots,N;\ j=1,2,\dots,(n_2+1)(m_2+1))$$

The function B(x,t) has been calculated according to relation (12) including the earlier determined coefficients A_{ks}. The found functions have been compared with the functions determined by the method of optimization. It was found that a strong smoothing is observed. This can be eliminated by taking logarithms in equation (19).

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